Pneumatic Tire Modeling by Membrane Method

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ABSTRACT

Tires play important roles in cars' performance. Since the early days of their production, many investigators have tried to find the relationship between tires' structural parameters and their performance. It has been established that mathematical methods are the most effective techniques, and are divided into different methods.

This paper deals with the thin shell theory from the membrane viewpoint and its related problems of stress analysis are discussed.

In further steps, the calculated results of a model for a four ply bias tire (7.00-16) are presented and compared with experimental data.

It is observed however that although this method is an approximate model and consists of many simplifying assumptions, the calculated results are in good agreement with experimental data.

INTRODUCTION

The pneumatic tire is a composite of rubber, textile cord and steel which are ingeniously combined to provide support of load, transfer of power, maintenance of direction, shock absorption, and riding comfort. It is comprised of parts or subassemblies which serve specific and unique functions. However in general it can be said that the tire consists of three main parts: tread, carcass and bead.

Tread is that part of a tire which must have contact with the road, so it is necessary to have both traction and wear resistance characteristics. Carcass is the heart of the tire and consists of textile cords (in some cases steel cords) interspersed with sufficient rubber to provide a cohesive matrix which is called ply. The bead, is composed of high tensile strength steel wire formed into inextensible hoops functioning as anchors for carcass plies, and holding the assembly on the rim of the wheel. Figure 1 shows a section from a tire. This type of tire is well known as the bias or conventional tire. The other types are radial and bias-belted tires which are out of the scope of this paper.

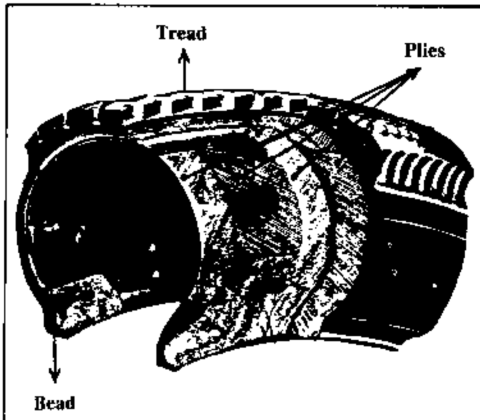

Figure 1: A schematic view of tire section [1].

Before taking the mathematical modeling of the tire into consideration, it is necessary to discuss briefly the tire constructing method. Bias tires are fabricated by the so called tire expansion process. Schematically illustrated in Figure 2, the carcass is

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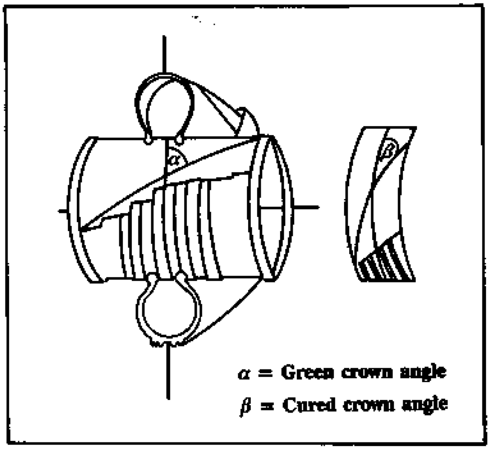

Figure 2:Tire construction and expansion [3].

built on a cylindrical drum by applying pieces of the layers or plies, whose lengths equal the circumference of the drum.The protective sidewall and tread rubber is then applied over the carcass.At this stage, the green tire has been made.To bring the tire from its present cylindrical shape into its final toroidal shape, the expansion process is utilized in which, by applying pressure into the green tire, the final shape will be produced.During the expansion process, cord angle α of the green (or cylindrical) tire changes to the angle β as illustrated in Figure 2.While the cord angle α is constant, the angle β of the toroidal or expanded tire is not constant but changes from point to point.The illustrative equation which relates the cord angle β to any points of the tire to its green angle α, called "cosine law" will be discussed in the next sections.

The simplest stress analysis approach from a theoretical standpoint is the calculation of the shape taken by the inflation pressure and the stress developed in an inflated but otherwise unloaded bias tire.The first rigorous solution to this problem was obtained by Purdy in the U.S. in 1928 [1]. However it was not until 1956, when Hofferberth published his results in Germany, that such work became readily accessible to tire engineers. Biderman's investigations dealing with the same subject appeared in 1957 in the Soviet Union [1].

All the above mentioned investigations are based on this assumption that the structure of tire obeys membrane laws.

The membrane method for tire stress analysis is based on a number of simplifying assumptions which are discussed in the next section.These assumptions can affect, to a great extent the reality of tire structure, but as it will be seen later the results of analysis by this method are in very good agreement with experimental data. Therefore, we accept this type of analysis, despite the fact that today, complicated techniques such as finite element methods are extensively being used in tire design.

Mathematical Modeling

The tire structure must be idealized before the construction of the mathematical model is initiated. This idealization is based on the following five assumptions [2].

a)The membrane is assumed to consist only of the cords, that is, the binding material and protective rubber are neglected.

b)The carcass is thin-walled which means the thickness of the carcass wall is small in comparison to its overall dimensions.This allows the use of the membrane theory of elasticity, since bending is neglected.

c)The plies of cords are coplanar.

d)The cords act as a net that is, they are flexibly linked at their intersection points.There is no relative motion between the cords at these points, but the deformation between the intersection is unrestrained.

e)The tire under consideration is symmetric, inflated but not loaded.

This pneumatic tire carcass with the neglected rubber may be visualized as a deformed membrane in the form of a surface of revolution, obtained by rotation of a plane curve about an axis in the plane of the curve.This curve is known as the meridian curve or the meridian shape of the tire (Figure 3).

From the theory of thin shells, the membrane equations can be deduced as follows [2]:

$$\frac{1}{r_\phi}\frac{\partial(T_\gamma)}{\partial\theta} + \frac{\partial N_\theta}{\partial\theta} + T\cos_\phi + P_\theta y = 0 \qquad (1)$$

$$\frac{1}{r_\phi}\frac{\partial(N_\phi y)}{\partial\phi} + \frac{\partial T}{\partial\theta} - N_\theta\cos_\phi + P_\phi y = 0 \quad (2)$$

$$\frac{N_\theta}{r_\theta} + \frac{N_\phi}{r_\phi} = P_v \quad (3)$$

Where N_θ=force in the direction of the tangent to the parallels per unit length, N_ϕ=force in the direction of the tangent to the meridians per unit length, r_ϕ and r_θ are principle radii of curvature, $T=N_{\theta\,\phi}=N_{\phi\,\theta}$=shearing force, P_θ=load component in the direction of the parallel circle tangent, P_ϕ=load component in the direction of the meridian tangent, P_v=load component normal to the surface (Figure 3) [2].

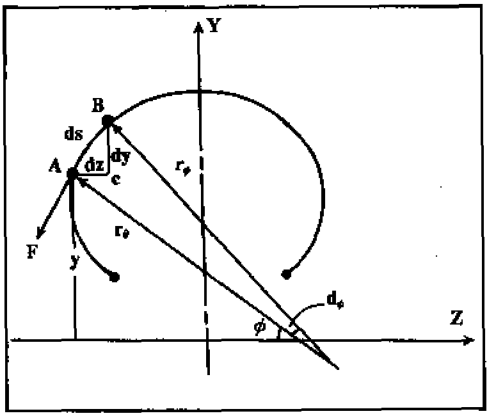

Figure 3:Tire meridian section curve with co-ordinates.

Principle Stresses and Cord Tensions

The stress generated in the membrane and the tensions in the cords can be calculated from the equilibrium equations (1), (2) and (3) utilizing the meridian equation.We assume that the meridian curve has the following equation (Figure 3):

$$Z = \int f(y)dy \quad (4)$$

where y, z represent the radial and lateral directions, respectively.The equation of f(y) will be derived later.Now we concentrate on solving the equilibrium equations.First of all, the following parameters should be defined.They are known as reduced stresses [3].

$$\bar{N}_\theta = (y'/y\cos_\phi)\,N_\theta \quad (5)$$

$$\bar{N}_\phi = (y\cos_\phi/y')\,N_\phi \quad (6)$$

Rewriting the equilibrium equations in terms of these reduced stresses and further assumptions, will give:

$$\frac{\partial\bar{N}_\theta}{\partial\theta} = 0 \quad (7)$$

$$\frac{\partial\bar{N}_\phi}{\partial z} - yy'P = 0 \quad (8)$$

$$\bar{N}_\theta - \frac{y''}{y}N_\phi - [1 + (y')^2]\,P=0 \quad (9)$$

These additional assumptions are:

- Because of symmetry the shearing forces must be vanished, i.e. $T = N_{\phi\,\theta} = N_{\theta\,\phi} = 0$

- It was assumed that the only external load is inflation pressure, so since the resultant force of internal pressure is normal to surface, it can be deduced, $P_\theta = P_\phi = 0$ and $P_v = P$, where P is the inflation pressure.

In the above equations, y' and y'' denote dy/dz and d^2y/dz^2 , respectively; y' also is equal to $\cot g_\phi$ as can be seen in Figure 3.

With substituting the following equation [3]:

$$\frac{\partial\bar{N}_\phi}{\partial z} = \frac{\partial\bar{N}_\phi}{\partial y}\cdot\frac{dy}{dz} = \frac{d\bar{N}_\phi}{dy}\cdot y' \quad (10)$$

into equation (8) and solving the new differential equation, one can obtain:

$$\bar{N}_\phi = \frac{Py^2}{2} + C \quad (11)$$

where the C is constant of integral.From Figure 3, it may be shown that, when $y = y_B$, then $f(y_B) = 0$, so from equation (6), it can be deduced:

$$y = y_B \qquad \bar{N}_\phi = 0 \quad (12)$$

Now, substituting equation (12) into equation (11), and the resultant relation into equation (6), we can write:

$$N_\phi = \frac{P}{2y}\frac{(y^2 - y_B^2)}{f(y)}\sqrt{1 + f^2(y)} \quad (13)$$

We can derive the same general relation for N_θ.If this is done, it can be deduced that:

$$N_\theta = \frac{Py\sqrt{1 + f^2(y)}}{f(y)}\left[1 - \frac{1}{2y}\frac{(y^2 - y_B^2)\,f'(y)}{f(y)}\frac{1}{1 + f^2(y)}\right] \quad (14)$$

These forces must be taken up by the carcass cords of the tire. For determining the cord tension in the tire, consider the point A in Figure 4. The

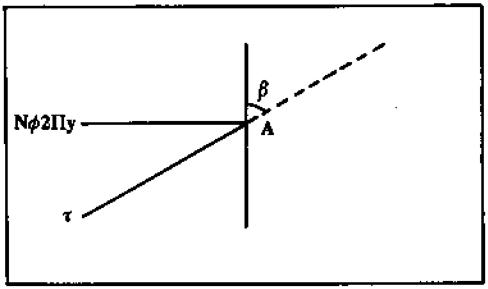

Figure 4: Diagram of forces acting at point A in Figure 3.

position of this point is the same as the point A in Figure 3. From the equilibrium condition, we can write:

$$\tau M \sin_{\beta} = N_{\phi} 2\Pi y \tag{15}$$

Where, τ = cord tension, M = total number of cords in tire and β = angle between the cords and circumferential direction. There are some relations for calculating β from y or radius of tire but the most common formula which is most generally used is the cosine law. This formula is [2, 3]:

$$\cos_{\beta} = \frac{y\cos_{\alpha}}{C_t r_s} \tag{16}$$

Where, α = a constant angle which is known as green angle, r_s = bead point radius, (Figure 3), C_t = average cord extension ratio.

It is common to express M, the total number of of cords in terms of n_0, or number of cord ends per unit length, which is a constant value in the green state. This expression is:

$$M = 2\,\Pi n_0 r_s \sin\alpha \tag{17}$$

Therefore, we can write the cord tension in the following form:

$$\tau = \frac{N\phi y}{n_0 r_s \sin\alpha \sin\beta} \tag{18}$$

With reference to Figure 3, one can derive the following expression for cord length and meridian profile length:

$$L = 2\int_{r_s}^{r_a} \frac{[1 + f^2(y)]^{1/2}}{\sin\beta}\,dy \tag{19}$$

$$L_m = 2\int_{r_s}^{r_a} [1 + f^2(y)]^{1/2}\,dy \tag{20}$$

Where L = cord length, L_m = profile length and r_a=maximum radius of tire profile at the crown.

There are also two parameters which can be derived from the membrane theory. They are bead tension force and shear stress applied from rubber matrix to the cords. These formulas are [4]:

$$T = \frac{(1/2)P\,(y_B^2 - r_s^2)}{f(r_s)} \tag{21}$$

$$U = \frac{P}{2f(y)}\,\frac{(y_\beta^2 - y^2)}{y^2}\,\frac{1}{\sin^2_{\beta}}\left[\cot g\beta + \frac{y\cos_{\alpha}}{C_t r_s \sin_{\beta}}\right] \tag{22}$$

Where T = bead tension force, y_B = radius of tire at maximum section width and U = shear stress. The value of shear stress at y = r_a or at the crown is equal to zero.

Determination of f(y)

It was seen that for computing the all parameters, one should know the f(y) or meridian equation. The general expression of meridian contour equation can be obtained by solving of an ordinary differential equation which is known as the shape equilibrium differential equation [2, 3]:

$$\mathrm{Cotg}\phi\; d\phi + \cot g^2\beta \frac{dy}{y} = \frac{2y}{y^2 - y_B^2}\,dy \tag{23}$$

This differential equation, is a special Bernoulli type [5]. This is a non-linear ordinary differential equation, tractable by two successive iterations. After solving this equation, we will have [2, 3]:

$$f(y) = \frac{(y^2 - y_B^2)\left[C_t^2 \frac{r_s^2}{\cos^2\alpha} - y^2\right]^{1/2}}{\left[(r_a^2 - y_B^2)^2 \left(C_t^2 \frac{r_s^2}{\cos^2\alpha} - r_a^2\right) - (y^2 - y_B^2)^2 \left(C_t^2 \frac{r_s^2}{\cos^2\alpha} - y^2\right)\right]^{1/2}} \tag{24}$$

By integration of this equation, one can find the desired formula for plotting the tire meridian contour, thus:

$$Z = \int f(y)\,dy \tag{25}$$

Now with the help of the above expression along with equation (24) one can compute all of the required parameters from the known values.

RESULTS AND DISCUSSION

Numerical Results

In this section, the obtained relations will be applied for a known size tire and the numerical results will be presented.

The size of the chosen tire is 7.00-16.The 4-ply bias tire's initial cord angle (or green angle) is 60° and the dimensions of the tire meridian curve are 225.019mm for bead point (drum radius), 227.992mm for radius at maximum section width and 359.992mm for radius at the crown or maximum radius of the tire.The average cord extension ratio is equal to 1.02 and the inflation pressure is 0.4137 MPa.

The calculations were performed at 21 points of the tire radius between bead point and crown. The value of n_0 is 4 (for 4 plies) per mm.

Figures 5, 6 and 7 show the principle forces per unit length in meridian and circumferential or parallel directions, cord tension and shear stress acting from rubber matrix to the cord, respectively.

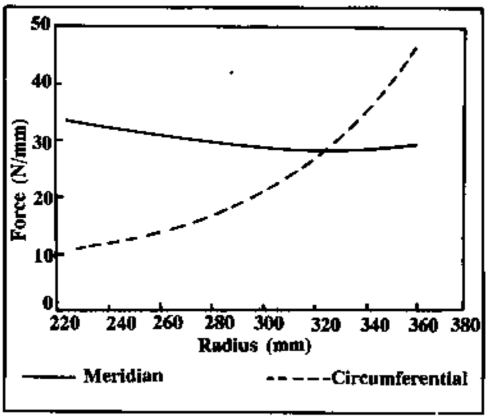

Figure 5:Principal forces

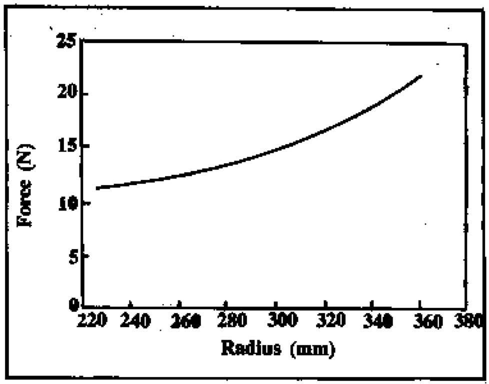

Figure 6:Cord force

In computing f(y), it can be seen that the value of function will be infinite at the crown or at $y = r_s$, so it is not possible to calculate the integral from ordinary methods of numerical computation such as trapozoidal or Simpsons.

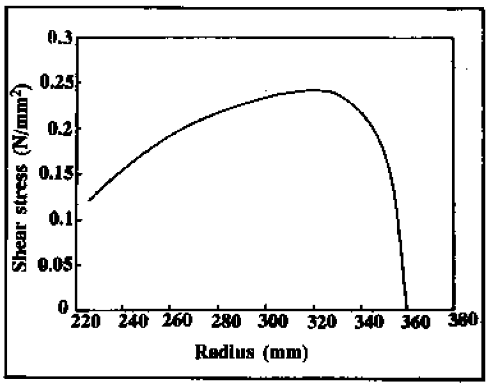

Figure 7:Shear stress

For determining the whole tire contour or meridian profile, it is necessary to integrate equation (24).The complicated shape of this integral necessitates computation from a numerical method, so we should write:

$$Z_{i+1} = Z_i + \int_{y_{i+1}}^{y_i} f(y)dy \qquad (26)$$

All investigators (for example see reference [3]), who have worked on the solution of this integral, used the mentioned methods with a slight difference.They used r_s-Δy instead of r_s for the upper limit of the integral, where Δy is a small quantity.The value of Δy can affect the results to a great extent, therefore the conventional methods of numerical integration are not suitable.To overcome this difficulty we have computed the integral from a well known numerical method which is called "Gaussian Quadrature" Technique [6].

The five points Gaussian integration method was used and the coordinates of tire meridian profile have been obtained. Figure (8) shows the tire meridian profile which was obtained by Gaussian numerical integration method.

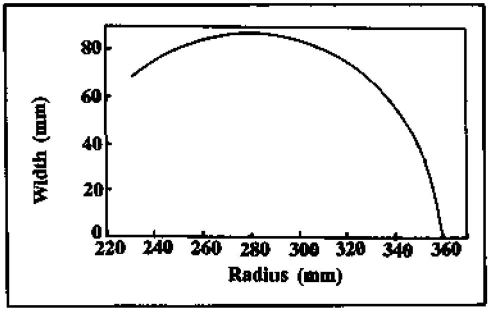

Figure 8: Inflated tire meridian profile

equilibrium shape of tire under inflation. Utilizing the later parameter, makes it possible to draw the inflated shape of a tire from the standard data, so it can be used in designing of new tires. Also it is shown the calculated equilibrium shape is in very good agreement with experimental data, therefore the users of this method can be certain that it can be applied to the tire design without any problem.

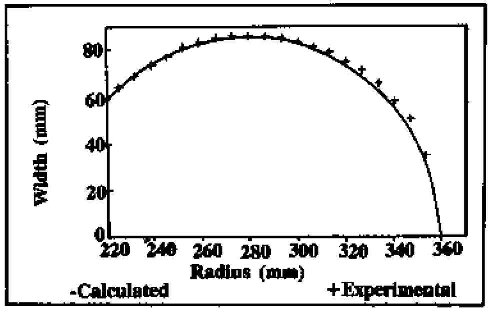

Figure 9: Comparison between calculated and measured inflated profile

Comparison with Experimental Results

In order to determine the ability of this mathematical model for simulation of the tire at its inflated shape, the internal inflated profile of the previous mentioned tire has been determined by subtracting the tire thickness from the external inflated profile. The latter shape has been measured by a profilemeter. Figure 9 shows the tire meridian profile which was obtained by numerical simulation and compares it with experimental data. It can be seen that the experimental results are in good agreement with calculated ones.

CONCLUSIONS

It is shown that by using the thin shell theory, the equilibrium equation for a bias tire under inflation can be deduced. Having solved these equations, one can compute the structural parameters of tires, such as, principle forces per unit length, cord force, shear stress acting from rubber matrix to cords, cord length, meridian length, bead tension and the

NOTATION

α green angle

β crown cured angle

τ cord tension force

C_t average cord extension ratio

N_θ force in direction of tangent to the parallel per unit length

N_ϕ force in direction of tangent to the meridian per unit length

$N_{\theta\phi}$, $N_{\phi\theta}$ shear force

P inflation pressure

P_θ load component in direction of parallel circle tangent

P_ϕ load component in direction of meridian circle tangent

P_v load component normal to the surface

M total number of cords

n_0 cord ends per unit length

L cord length

L_m profile length

r_A maximum or crown radius

r_e drum or bead point radius

r_θ, r_ϕ principle radii of curvature

T shear force

T bead tension force

U shear stress

y tire radius

y_B radia at maximum section width

z tire width

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